

The miniature, reversed Stirling cycle cryo-cooler: integrated simulation of performance

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Abstract

Miniature cryogenic coolers operating on the reversed Stirling cycle are used for cooling the detectors of thermal imaging systems. The type is a candidate for duty in satellites—a singularly demanding environment, since whatever the cooling power achieved, there is pressure for it to be greater; whatever the electrical power consumption, there is demand for it to be less. Stirling cycle machines, both prime movers and reversed-cycle, are renowned for the high cost of trial-and-error development. The situation calls for improved understanding of the cycle of thermodynamic processes. This paper makes a start by breaking with the tradition of arbitrary linearisation of the defining equations, and by integrating the characteristics of the electromagnetic drive circuit with simultaneous computation of the dynamic, thermodynamic and fluid flow processes. A specific cooler is defined and simulated performance points adduced in evidence of correct functioning of the computations. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Stirling cycle; Regenerator; Thermal regenerator; Cryo-cooler; Stirling cycle simulation

Notation

$a(1,1)$, $b(1)$ coefficients of array –
etc.

a , b , aa , bb constants defined in text ()

A_{ff} free-flow cross-sectional area m^2

A_w wetted area m^2

A_x cross-sectional area exposed to gas
pressure m^2

C_f friction factor-equivalent wall shear stress
divided by $1/2\rho u^2$ –

COP coefficient of performance—ratio heat
lifted/work in –

D cylinder bore m

F $1/2C_f u^2/r_h$ m/s^2

F_{sol} force generated by solenoid N

F'_{sol} $F_{sol}/k_p V_{sw}^{1/3}$

f cyclic frequency s^{-1}

h coefficient of convective heat transfer
 W/m^2K

H enthalpy J

k thermal conductivity W/mK

L_r length of regenerator packing m

L_{ref} reference length ($= V_{sw}^{1/3}$) m

M total working fluid mass kg

m variable mass kg

m' mass rate kg/s

N_{MA} speed parameter or characteristic Mach
number $\omega V_{sw}^{1/3}/\sqrt{RT}$ –

N_{pr} Prandtl number, $\mu c_p/k$ –

N_{RE} characteristic Reynolds number

$N_{MA}^2 N_{SG}$ –

N_{re} local, instantaneous Reynolds number

$4\rho u r_h/\mu = 4r_h m'/\mu A_{ff}$ –

N_{SG} Stirling number $p_{ref}/\omega\mu$ –

p absolute pressure Pa

q , q' heat, heat rate J , J/s

P , Q abbreviations for terms arising in the
integrating factor method of solution of a
first-order differential equation (definitions
in text) (–)

Q' volume flow rate m^3/s

r_h hydraulic radius—free-flow area/wetted
perimeter m

R gas constant for specified gas J/kgK

R_e Annand's Reynolds number, $2fSpD/\mu$ –

S stroke m

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s	specific entropy J/kgK
T	absolute temperature K
t	time s
U	internal energy J
$U()$	step function defined in text –
u	bulk velocity, $m'/\rho A_{ff}$ m/s
$\underline{u}$	dimensionless u defined $u/\omega L_{ref}$ –
V_E, V_C, V_{buff}	volume excursion of expansion/compression/buffer space m^3
V_{sw}	swept volume m^3
W	work J
Y_{sep}	vertical distance separating displacer from work piston at rest m
Y_{ampd}	peak allowable amplitude (semi-stroke) of displacer m
Y_{ampp}	peak allowable amplitude (semi-stroke) of work piston m
y_d	vertical excursion of displacer from piston datum (rest) position m
δ	dimensionless volume—usually dead volume— V_d/V_{sw} –
μ	coefficient of dynamic viscosity Pas
ν_D	dead space parameter ($= \delta_i \ln N_T / (N_T - 1)$) –
ρ	density kg/m^3
σ, σ'	dimensionless mass, m/M , dimensionless mass rate, $m'/\omega M$ –
τ	dimensionless temperature, T/T_C , e.g. $\tau_g = T_g/T_C$ –
ω	angular frequency rad/s
η_v	regenerator volumetric porosity (pore volume/envelope volume) –

Subscripts

ag	air gap
buff	buffer space or 'crank'-case
E,C	Expansion (temperature), Compression (temperature)
c,e,r	compression, expansion, regenerator
d	dead (unswept) volume displacer
f	(due to) friction
ind	indicated (as opposed to brake)
p	piston
q	(due to) heat transfer
ref	(constant) reference value
rms	root mean square
rod	relating to displacer drive rod
sol	solenoid
sw	swept (volume)
w	wetted (area)

1. Introduction

The Stirling cycle cooler operates on a closed, regenerative cycle: closed because there are no valves, a fixed

mass of working fluid being cycled repeatedly through the same thermodynamic sequence; regenerative on the strength of the key, central component, the thermal regenerator. Comprising nothing more than a porous, conducting matrix with a high ratio of wetted area to volume, this passive, deceptively simple component stores and releases heat internally which would otherwise be exchanged with the surroundings at temperatures intermediate to the cycle limits of T_E and T_C . To this extent, the closed regenerative cycle fulfils the basic condition for thermodynamic reversibility.

The cycle of operations is illustrated in Fig. 1, which shows in schematic form a cooler of coaxial configuration. A work piston controls total volume, while displacer and work piston in conjunction determine the distribution of the working fluid between expansion and

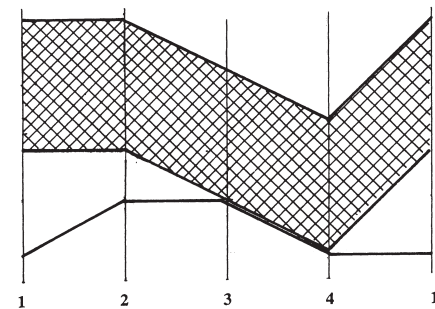
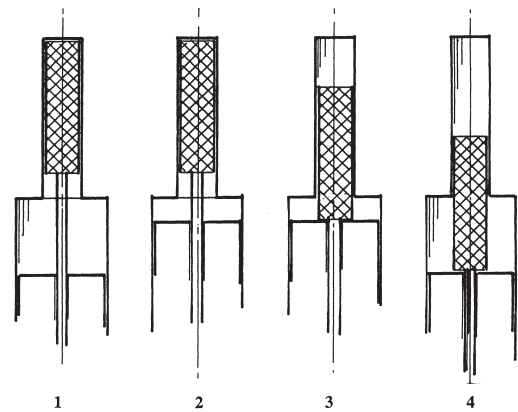


Fig. 1. Schematic representation of the sequence of volume changes of the Stirling cycle. The sequence is common between engine (prime mover) and heat pump (refrigerator) modes.

compression spaces. The regenerator, frequently comprising a stack of fine wire gauzes, fills the hollow displacer and forms the flow path between the two variable-volume spaces.

A typical cycle may be thought of as starting with the piston at the outer dead centre, and the displacer at the uppermost limit of its travel. The work piston rises, bringing about a compression of the entire working fluid mass, a rise in temperature overall and a corresponding overall rejection of heat. The majority of the working fluid mass being in the compression space during this process, the majority of the heat rejection occurs at or near compression space temperature, T_C . The displacer is now caused to fall, and in the process brings about a transfer of a large fraction of the working fluid via the regenerator to the expansion space. The passing fluid deposits much of its heat in the regenerator, emerging close to expansion temperature, T_E . Both piston and displacer now move downwards. The overall expansion lowers the temperature of the fluid occupying the expansion space. For as long as this temperature remains below that of the expansion space walls, the fluid absorbs heat, bringing about the required cooling effect. Towards the end of the expansion phase, the displacer is caused to rise, transferring working fluid via the regenerator back to the compression space. In the case of an appropriately designed regenerator, the fluid picks up on this reverse pass the majority of the heat deposited in the matrix on the previous (hot–cold) pass, emerging with a temperature close to T_C .

The reversed Stirling-cycle cooler is capable of operating over a range of temperature differentials, and has been produced commercially in a wide range of heat-lift capacities. Miniature variants (Fig. 2) tend to operate between 77 K and ambient, lifting a fraction of a watt for an electrical input of a few watts. This impressive coefficient of performance has been achieved largely by driving the work piston by linear electro-magnetic excitation, thereby eliminating side thrust and resulting friction. The displacer is suspended on a mechanical spring of some type and actuated by the pressure differences arising during the working cycle.

The cycle of work, heat exchange and fluid flow variations is a complex dynamic and thermodynamic interaction. Design of coolers from first principles calls for this interaction to be understood—a challenge traditionally approached by linearising the various disturbing and restoring forces on piston and displacer and treating the whole as a 2-degrees-of-freedom system. In the hands of de Monte and Benvenuto, the approach has been taken to a high art form [1].

On the other hand, it can be argued that most of the relevant phenomena are, in reality, non-linear. While linearisation of an isolated, non-linear phenomenon may frequently be justified, no-one knows for certain how good a representation of reality is achieved by arbitrary

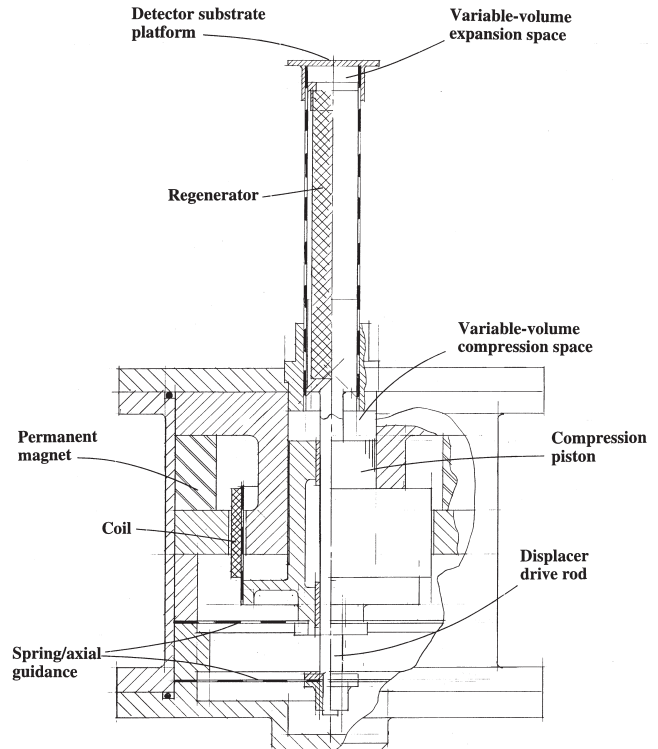


Fig. 2. Slightly simplified cross-section through 'B-Type' cooler. Loudspeaker suspension is indicated for both piston and displacer. To gain full benefit from this approach to suspension and guidance it would probably be necessary to specify a 'bucket' piston suspended at both ends. Reproduced with the permission of DERA, Malvern.

linearisation of the interaction of many non-linear effects. But perhaps the strongest case for a return to the basics of Stirling cooler dynamics is that the mathematical contortions to which linear algebra leads assume greater complexity than a simple head-on solution of the original, non-linear problem, and the process involves an uncomfortably high ratio of algebra to insight.

This paper accordingly adapts a well-known cycle analysis [2] to the case. Equations of motion for piston and displacer are written in terms of instantaneous gas, spring and drive solenoid forces. An implicit numerical integration scheme follows the gas processes and piston motions from start-up to cyclic equilibrium, revealing aspects of dynamic behaviour suppressed by the linear approach. Basic to the re-formulation of the problem are the equations of motion of displacer and work piston.

2. Mechanical equations of motion

The relevant features of the mechanical system are represented with associated notation in Fig. 3(a). Piston motion, y_p , refers to the upper piston face, and the datum is piston rest position. The same datum is used for displacer motion, y_d , which refers to the lowermost face of the regenerator housing. Amplitudes (semi-strokes) of

increment of time dt is considered corresponding to increment $d\phi$ in dimensionless time (or ‘crank’ angle). Energy conservation for the space may be expressed as

$$dQ + dH - dW = dU$$

In general, m changes by dm in the interval: if mass enters it does so from the regenerator and at the relevant regenerator outlet temperature, T_{ro} . When dm leaves, on the other hand, it does so at the instantaneous value of the varying T_g . Following Finkelstein [2], step function $U(dm)$ is defined such that:

$$\begin{aligned} U(dm) &= 1 \text{ for } dm \geq 0 \\ &= 0 \text{ for } dm < 0 \end{aligned}$$

This permits the effects of dm to be handled in a single algebraic term. With h for instantaneous coefficient of convective heat transfer and A_w for the corresponding wetted area, the energy balance may be written as

$$\begin{aligned} hA_w(T_w - T_g) dt + c_p dm \{T_g[1 - U(dm)] \\ + U(dm)T_{ro}\} - p dV = c_v(T_g dm + m dT_g) \end{aligned} \quad (6)$$

3.2. Mass conservation

On the basis of a linear variation of temperature between T_E and T_C and a small pressure gradient, the instantaneous mass content of the regenerator is $m_r = 1/2(p_e + p_c)V_{dr} \ln N_T / \{RT_C(N_T - 1)\}$ [2,3]. Mass inventory, M , of the gas circuit being assumed constant, the sum $m_e + m_c + m_r = M$, may be notionally differentiated to give

$$dm_e + dm_c + \frac{V_{dr}}{RT_C} \frac{\ln N_T}{N_T - 1} 1/2(dp_e + dp_c) = 0 \quad (7)$$

3.3. Gas law

An advantage of formulating in terms of a matrix of unknowns is that it is unnecessary to make algebraic substitutions in terms of working fluid properties from the gas law. Fluid temperature falls *below* nominal T_E during expansion so that, depending on the gas, conditions may become sufficiently close to the critical point to warrant consideration of an equation such as that of van der Waals.

For convenience of presentation the treatment proceeds in terms of the ideal gas equation, but will be seen to be capable of ready adaptation to more sophisticated models.

The gas law ($pV = mRT$) is differentiated, the result applying to either working space:

$$RT_g dm + Rm dT_g - V dp = p dV \quad (8)$$

The dV term is on the right-hand side, reflecting the fact that the dV are known via Eqs. (3) and (4) from the equations of motion.

3.4. Pressure drop in terms of flow resistance

The steady-flow component contributes the majority of the pressure drop across the regenerator. A severe temperature gradient and a substantial phase shift between the flows at the two extremities suggest integrating local pressure drops over elemental lengths, Δx , of regenerator packing to form instantaneous overall pressure difference. To achieve this in terms of the variables of the implicit solution, namely, the two pressures, p_e and p_c , and the corresponding mass rates, m'_e and m'_c , calls for a bit of sleight of hand.

The usual expression for steady-flow pressure drop takes a negative sign (flow *down* the pressure gradient). The present treatment requires $p_e - p_c$ to be positive when flow is from expansion space to compression. With reference to Fig. 3(b):

$$p_{j+1} - p_j = + 1/2 \rho u^2 C_f \frac{dx}{r_h} \frac{u}{|u|}$$

The term $u/|u|$ keeps the higher pressure upstream of the lower. The expression may be re-written in terms of local $m' = \rho u A_{ff}$. With $\rho = p/RT$ and with $x/L_{reg} = \lambda$

$$p_{j+1} - p_j = \frac{m'^2}{p_e + p_c} \frac{C_f L_{reg} RT}{r_h A_{ff}^2} \frac{u}{|u|} d\lambda$$

At any given time, m' varies with location, x , between m'_e and m'_c , and over substantial intervals of crank angle may be inwards from both ends of the regenerator or outwards from both ends. An expression for instantaneous m' at location λ which accommodates all such flow possibilities but with some compromise at locations intermediate to the two ends is:

$$\begin{aligned} m'(\lambda) &= -m'_e + (m'_c + m'_e)\lambda \\ &= m'_e(-1 + \lambda) + m'_c\lambda \end{aligned} \quad (9)$$

Eq. (9) reflects the fact that positive m'_e is flow *into* the expansion space, and therefore flow *out* of the expansion end of the regenerator.

The customary approximation to local temperature is linear:

$$T(\lambda) = T_E + (T_C - T_E)\lambda \quad (10)$$

Taking out constant factors, noting that m' is dm/dt , and evaluating pressure drop at mid-interval dt :

$$p_e + 1/2 dp_{j+1} - p_c - 1/2 dp_j = \frac{L_{\text{reg}}}{r_h} \frac{R}{p_e + p_c} \frac{1}{A_{\text{ff}}^2} \frac{(dm_e(-1 + \lambda) + dm_c \lambda)}{dt^2} \frac{dm}{|dm|} \dots \\ \dots C_f(T_E + (T_C + T_E)\lambda) d\lambda$$

Summing over the length of the regenerator gives an expression which may be made quasi-explicit in the dm_e and dm_c as well as in the pressure increments:

$$1/2 dp_e - 1/2 dp_c = \frac{L_{\text{reg}}}{r_h} \frac{R}{p_e + p_c} \frac{1}{A_{\text{ff}}^2} \frac{d\lambda}{dt^2} \{dm_e[dm_e \Sigma_1 + dm_c \Sigma_3] \dots + dm_c[dm_e \Sigma_2 + dm_c \Sigma_3]\} - p_e + p_c \\ \Sigma_1 = \sum_{j=1}^n (-1 + \lambda)^2 C_f(T_E + (T_C + T_E)\lambda) dm/|dm| \\ \Sigma_2 = \sum_{j=1}^n \lambda^2 C_f(T_E + (T_C + T_E)\lambda) dm/|dm| \quad (11) \\ \Sigma_3 = \sum_{j=1}^n \lambda(-1 + \lambda) C_f(T_E + (T_C + T_E)\lambda) dm/|dm|$$

In evaluating the Σ the C_f and $dm/|dm|$ corresponding to local, instantaneous m' and T (Eqs. (9) and (10)) are used to form the local, instantaneous Reynolds number, N_{re} . The corresponding friction factor, C_f , then follows from an interpretation of the Kays and London correlation [7] for the appropriate value of volumetric porosity, $\mathfrak{P}_v$. For the $\mathfrak{P}_v$ of the specification:

$$C_f = 40/N_{\text{re}} + 0.25 \quad (12)$$

3.5. Regenerator temperature distribution

Until 5 years ago, temperature solutions for regenerator response to the flow and pressure conditions of the Stirling machine were unavailable. This has changed since development of a 'natural coordinate' system [3] for creation of the coincident Lagrange/Euler integration grids required for simultaneous solution of fluid and matrix temperatures. Acquiring a comprehensive picture of regenerator in situ transient response for the crank-driven Stirling cycle machine is now a routine matter. On the other hand, it calls for generation of a fluid particle trajectory map for each set of operating conditions. While the computational effort was not great for the original formulation [3], the generalisation of 'natural' coordinates called for by the arbitrary volume and temperature variations of the present treatment [4] require significantly more numerical processing—hence the search for a suitable simplified approach.

Thermal capacity ratio, N_{TCR} is defined in terms of wire properties, ρ_w , c_w as $\rho_w c_w T_{\text{ref}}/p_{\text{ref}}$. The numerical

value in the present instance is 1250. This is twice that of the Philips MP1002CA air engine, which in turn exceeds that of the GPU-3 engine by a factor of almost 5. This, in conjunction with the massive volume of regenerator material in the cooler relative to that of the prime movers, suggests provisionally ignoring matrix temperature swing as insignificant in relation to fluid temperature fluctuations. Actual values are computed later.

With matrix temperature invariant, the variation of fluid/wall temperature difference, $\Delta\tau = (T_g - T_w)/T_{\text{ref}}$, along a particle path is given [3] by

$$\frac{D\Delta\tau}{d\phi} + P(\phi)\Delta\tau = Q(\phi) \quad (13)$$

In Eq. (13)

$$P = |u|N_{\text{st}} \frac{L_{\text{reg}}}{r_h} = |u|NTU$$

The NTU are evaluated via N_{st} corresponding to the local instantaneous Reynolds number, N_{re} , using the Kays and London [7] correlation appropriate to the value of volumetric porosity, $\mathfrak{P}_v$

$$N_{\text{st}}N_{\text{pr}}^{2/3} = a/N_{\text{re}}^b \quad (14)$$

Q takes account of the pressure-swing term:

$$Q = -\frac{u}{\psi} \frac{\partial \tau_w}{\partial \lambda} + \frac{\tau_w}{\psi} \frac{\gamma - 1}{\gamma} \frac{\partial \psi}{\partial \phi}$$

The analysis is presented in full in [3]. For an interval over which flow may be considered steady, the solution of Eq. (13) proceeds in steps starting at the inflow end. The first step sets $\Delta\tau$ to inlet temperature difference:

$$\Delta\tau_{\phi + \Delta\phi} = Q/P + (\Delta\tau_{\phi} - Q/P)e^{-P\Delta\phi} \quad (15)$$

Inlet temperature difference, $\Delta\tau_{\text{in}}$, is established, by definition, at a time *previous* to that of the current computational step, and so is always known regardless of flow direction. The only subtlety in the computation is that it is necessary to 'step back' by the appropriate angular amount of dimensionless time, ϕ , for the $\Delta\tau_{\text{in}}$ appropriate to calculation of the $\Delta\tau$ at given location λ .

The solution for $\lambda = 1$ is compression space outlet temperature difference, that for $\lambda = 0$ the expansion-end outlet difference. These provide the working space *inlet* temperatures, T_{ro} , required for evaluation of Eq. (6). Eq. (15) also provides the $\Delta\tau$ intermediate to the two ends for plotting the temperature envelopes and reliefs discussed later. The NTU on which the P are based are

calculated in terms of the Reynolds number, N_{re} , corresponding to local instantaneous flow conditions.

The approximation improves as regenerator dead space ratio reduces and flow at outlet falls increasingly in phase with that at inlet, i.e. as steady flow conditions are approached.

3.6. Instantaneous heat transfer coefficient in the variable-volume spaces

Values are required of the coefficient of convective heat transfer, h , for use in the energy equations (Eq. (6)). The task is best done in terms of Stanton number, N_{st} , correlated in turn with Reynolds number, N_{re} (and Prandtl number, N_{pr}) for conditions in the variable-volume spaces. The only resource known to the writer is Annand's correlations [5] for conditions in the cylinder of the reciprocating internal combustion engine. Annand determined values of a and b for the following expression for heat rate, q' :

$$q' = A_w \left\{ a \frac{k}{D} R_e^b (T_w - T_g) + c(T_w^4 - T_g^4) \right\}$$

in which the Reynolds number, R_e , is defined as

$$R_e = 2fS\rho D/\mu$$

Neglecting the radiation component as being insignificant at the temperatures involved, and defining h as the ratio of heat rate, q' , to the product of wetted area, A_w , with temperature difference, $T_w - T_g$, leads to

$$h = \frac{akR_e^b}{D} \quad (16)$$

Normalisation of the energy equation (Eq. (6)) yields a term $hA_w/\omega Mc_p$, which is recognizably NTU—although not yet in useful form. However, multiplying numerator and denominator by $|u|L_{ref}m\omega$ brings about a conversion to $NTU\sigma|u|$, where there is now the 'proper' $NTU = N_{st}L/r_h$. It remains to define hydraulic radius, r_h , for the working spaces.

r_h is defined as wetted volume, V_w , divided by wetted area, A_w . When exposed cylinder wall height is h , V_w for the expansion space is $1/4\pi D^2h$, while corresponding A_w is $2\{1/4\pi D^2\} + \pi Dh$. Forming the ratio, simplifying and normalising:

$$r_h^{-1} = 2/h/L_{ref} + 4/D/L_{ref} \quad (17)$$

r_h for the compression space is dealt with in a similar way. For consistency with Annand's definition of Reynolds number, dimensionless velocity, u , required for both

working spaces is ωS normalised by $\pi\omega L_{ref}$, or $S/\pi L_{ref}$, where S is stroke—of displacer or piston as appropriate.

4. Preparation for solution

When Eqs. (6) and (8) are written for each of the two working spaces, the resulting total of six equations are, but for the U terms, linear in the six unknowns, dm_e , dm_c , dp_e , dT_{ge} , dT_{gc} , and dp_c (in that order). For generality, computation proceeds in terms of normalised variables and dimensionless parameters. Appendix B carries out the normalisation and lists the coefficients of the unknowns.

Individual solutions (increments in the dependent variables) are added to respective values of σ_e , σ_c , ψ_e , τ_{ge} , τ_{gc} and ψ_c (in that sequence) at each integration step to form values corresponding to new crank angle $\phi + d\phi$. The formulation is free of division by terms which can take a value of zero, and consequently there are no problems with singularities. When $\phi = 2\pi$, values are compared with those for $\phi = 0$, as many cycles being repeated as are necessary for convergence.

5. Discretisation and normalisation of motion equations

Formulation of the numerical solution proceeds by forming a finite-difference approximation to the differential coefficients of the motion equations. For example

$$\frac{d^2y_p}{dt^2} \approx \frac{\Delta y_p^{new} - \Delta y_p^{old}}{\Delta t^2} \quad (18)$$

In this expression the increment of motion Δy_p^{new} is the 'unknown' to be determined by solution of the complete set of simultaneous difference relationships. Piston dynamic response may be thought of as being centred on current time, t (or dimensionless time or crank angle, $\phi = \omega t$) as suggested by the inset in Fig. 3(a): the forces responsible for motion (spring, pressure difference, solenoid drive force) are those representative of the interval, i.e. mid-interval values. Given that solution is proceeding in the sequence $i - 1$ to unknown $i + 1$ via i , mid-interval values are 'known'. This leaves Δy_p^{new} as the sole unknown in the motion equation, which is therefore explicit in that unknown, and does not require to be part of the matrix of the unknown thermodynamic increments, dm_e , dm_c etc.

The same normalising parameters are applied to the motion equations as to the gas process model, and the unknown in each case made explicit. With λ_p for dimensionless piston excursion, y_p/L_{ref}

$$d\lambda_p^{\text{new}} = d\lambda_p^{\text{old}} + \frac{\Delta\phi^2}{(\omega/\omega_{\text{np}})^2} \left\{ F'_{\text{sol}} - \lambda_p + \frac{p_{\text{ref}} L_{\text{ref}}}{\psi k_p} (\psi_b - \psi_c)(\alpha_{\text{xc}} - \alpha_{\text{xrod}}) \right\} \quad (19)$$

In Eq. (19) F'_{sol} is an abbreviation for the mid-interval (normalised) solenoid driving force derived in Appendix A. The corresponding expression for the increment in displacer motion is

$$d\lambda_d^{\text{new}} = d\lambda_d^{\text{old}} + \frac{\Delta\phi^2}{(\omega/\omega_{\text{np}})^2} \left\{ -\lambda_d + \frac{p_{\text{ref}} L_{\text{ref}}}{\psi k_p} [(\psi_c(\alpha_{\text{xc}} - \alpha_{\text{xrod}}) - \psi_e \alpha_{\text{xd}} + \psi_b \alpha_{\text{xrod}})] \right\} \quad (20)$$

Evaluating Eqs. (19) and (20) in advance of solving the 6×6 matrix of ‘thermodynamic’ unknowns provides values for the right-hand sides (volume changes) of Eq. (4)a and b by simple differentiation of Eqs. (3) and (4) for the respective volumes.

6. Specimen simulated performance

The writer has permission for the reproduction of Fig. 2 of the DERA B-type cooler, whose excitation frequency of 72 Hz is unclassified information. Pending specific permission to disclose dimensions, masses, etc., the entries in Table 1 are consistent with, and believed to be achievable in, the cooler as shown, but not, except by coincidence, part of the official specification. A woven-wire mesh regenerator is assumed. A wide range of relevant gauzes is manufactured by Messrs. G. Bopp and Co. [6] and can be supplied punched to size. Assuming rectangular, square weave justifies use of friction factor–Reynolds number ($C_f N_{\text{re}}$) correlations from the Kays and London compendium [7]. There is no attempt to ‘grade’ the packing despite the marked variation in physical properties and flow conditions between hot and cold ends, although the option has afforded the basis of an interesting study [8].

The cooler itself is the result of a programme of experimental development work unconnected with the current theoretical study. It is therefore of interest to see what, if anything, the latter has to say about the former.

The cooler is ‘started up’ with expansion and compression space temperatures, T_E and T_C pre-set to ‘rated’ values (Table 1). The starting condition for the regenerator is a distribution of matrix and fluid temperature which is linear between T_E and T_C . Fig. 4 shows indicator diagrams: the smallest trace is that for the expansion space, the largest for the compression space. The rightmost loop is the net indicator diagram.

Table 1

Selection from the data forming specification and operating point which produced Figs. 4, 5, 7–9. All items are believed to be achievable in the cooler illustrated in Fig. 2. However, with the exception of excitation frequency, f , which is cited with the permission of DERA Malvern, all ‘data’ are assumed, and do not represent the official specification of any cooler

Working fluid	helium	
p_{ref} charge press	8	bar
B	2.0	T(esla)
f excitation freq.	72*	cps
k_d	5218	N/m
k_p	841	N/m
m_d	8.0E – 03	kg
m_p	25E – 03	kg
N	125	solenoid turns
T_C	300	K
T_E	77	K
V_{sw}	2.13	cm ³
<i>Derived data</i>		
$A_{\text{ref}} (= V_{\text{sw}}^{2/3})$	1.65E – 04	m ²
$\delta_e (= V_{\text{de}}/V_{\text{sw}})$	0.05	(nom)
$\delta_d (= V_{\text{dr}}/V_{\text{sw}})$	0.447	
$\delta_c (= V_{\text{dc}}/V_{\text{sw}})$	0.05	(nom)
$L_{\text{ref}} (= V_{\text{sw}}^{1/3})$	0.0128	m
$L_{\text{ref}}/L_{\text{ref}}$	4.8	–
N_{MA}	0.0073	–
N_{SG}	1.04E + 08	–
r_h/L_{ref}	0.003	–
$N_T = T_E/T_C$	0.256	–
N_{TCR}	1250	–
$\omega/\omega_{\text{np}}$	1.0	–
$\omega/\omega_{\text{nd}}$	1.4	–
η_v	0.686	–

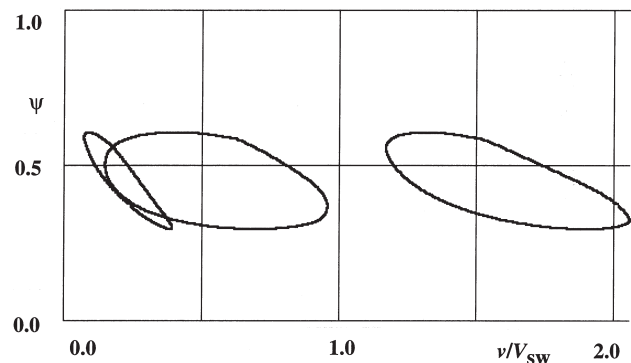


Fig. 4. Specimen computed indicator diagrams. The smallest, leftmost is for the expansion space. The larger trace at the left is for the compression space. The rightmost loop is the overall indicator diagram, and represents indicated work per cycle.

Fig. 5 shows computed variation of working space temperatures, τ_{ge} and τ_{gc} over a cycle ($0-2\pi$ rad). Relatively small swings either side of nominal cycle temperature limits confirm that high values of NTU_e and NTU_c are being achieved at ‘rated’ operating frequency and charge pressure chosen. The swings in temperature are

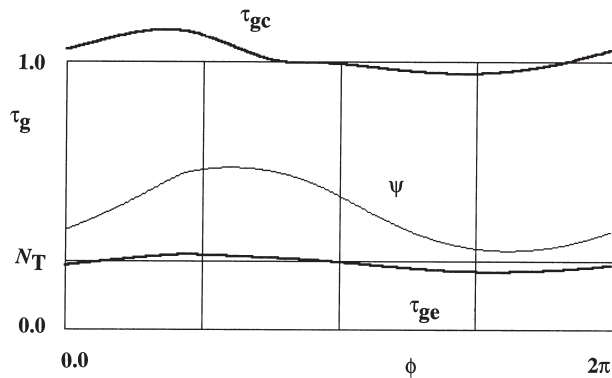


Fig. 5. Cyclic variation of dimensionless working space temperatures, T_{ge}/T_C and T_{gc}/T_C . Note the very small swings, consistent with the high rms values of ‘number of transfer units’ (NTU) for the exchange processes in the variable-volume spaces.

in phase with those of pressure (dashed curve), reflecting an element of adiabatic heating. The lower (expansion) trace is biased *below* nominal expansion-space temperature of N_T , consistent with heat flow *to* the working fluid. The upper trace is biased *above*, consistent with heat rejection at sink temperature.

Computed regenerator NTU are so high that the fluid temperature envelope plots as a straight line between cycle limits. To confirm that the temperature solutions reflect the picture given by more comprehensive treatments (e.g. those of [3]) NTU have been arbitrarily reduced by a factor of 50 for Fig. 6, in which (a) is a set of superimposed fluid temperature envelopes for a complete cycle at cyclic equilibrium, and (b) is the corresponding fluid temperature relief.

Fig. 7 shows normalised mass flow rate in function of ‘crank’ angle at opposite ends of the regenerator. The larger amplitude of the expansion trace is consistent with the greater fluid density, but neither trace bears much likeness to mass-rate records which have been computed for Stirling prime-movers, such as those of Creswick [9].

Fig. 8 plots the cyclic variation of Reynolds number, N_{re} , at the regenerator extremities. The N_{re} are corrected for the temperature dependence of coefficient of dynamic viscosity, μ , by use of the Sutherland Law [3]. The substantially larger values in the expansion space reflect high density (in the numerator of N_{re}) and reduced coefficient of dynamic viscosity (in the denominator).

In Fig. 9 the lowermost, light trace is driving voltage, which leads piston motion (lower heavy trace) by about 45° . The central light trace is pressure difference. The upper light trace plots expansion space pressure variation. The upper heavy trace is the lowermost face of the displacer. At the operating condition modelled, the amplitude of the regenerator pressure drop is about one third of expansion space pressure excursion. In a Stirling prime mover this ratio would be excessive and would probably prevent operation anyway. In the cooler as modelled it represents the cyclic interaction necessary

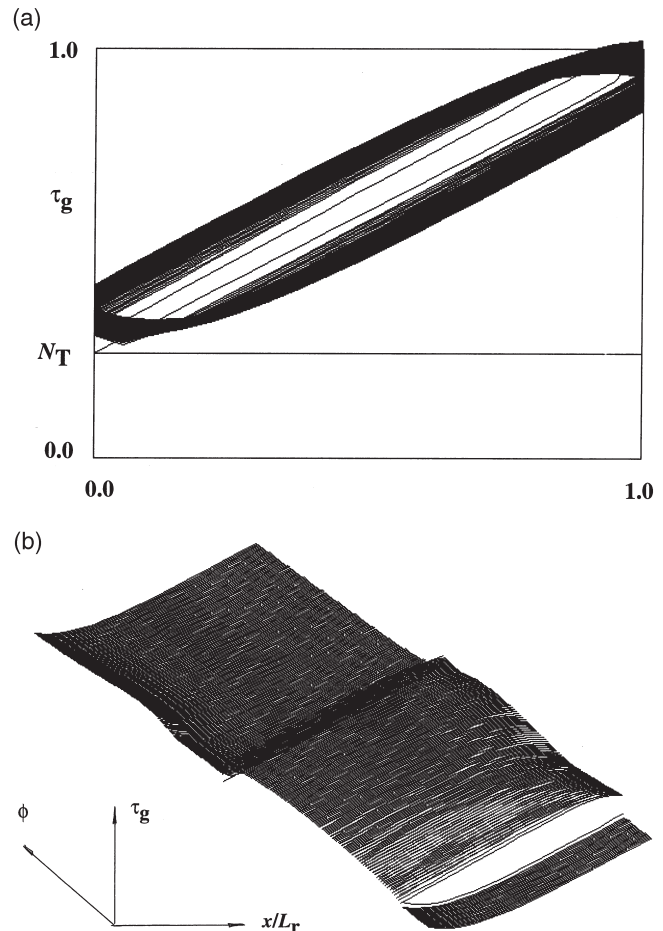


Fig. 6. Fluid temperatures computed from simplified regenerator theory for comparison with comprehensive solutions (see, e.g. [3]). (a) Temperature envelopes for working fluid occupying the regenerator, exaggerated by decreasing local, instantaneous NTU by a factor of 50. The machine fails to lift heat with the regenerator ‘modified’ in this way—as reflected by the fact that the temperature envelope lies permanently *above* the lower limit of the cycle, i.e. above N_T (heat exchange within this component is so effective that to obtain this plot it was necessary to decrease computed NTU for each integration step by factor of 50). (b) Fluid temperature relief corresponding to (a) above, i.e. NTU decreased by a factor of 50 to exaggerate temperature excursions.

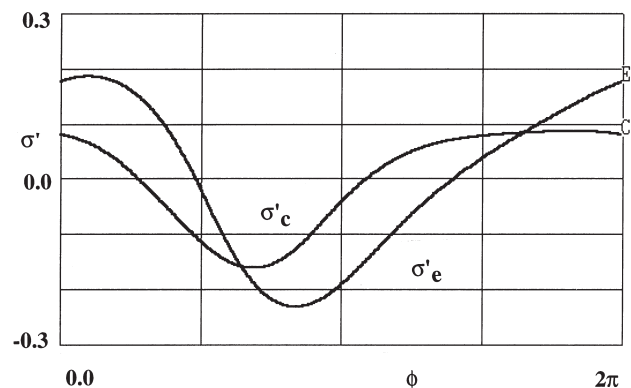


Fig. 7. Computed variations of dimensionless mass rate, σ'_e and σ'_c .

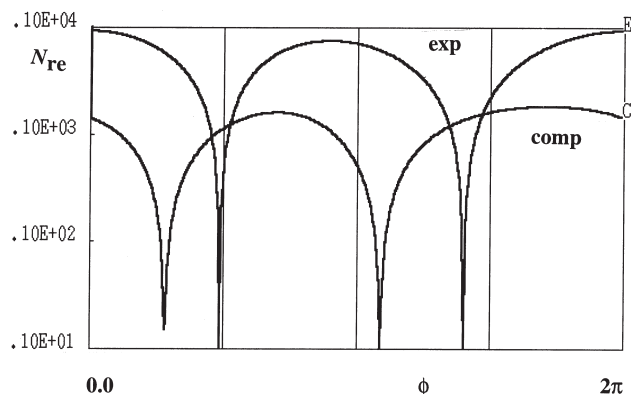


Fig. 8. Computed variations of Reynolds numbers at regenerator extremities.

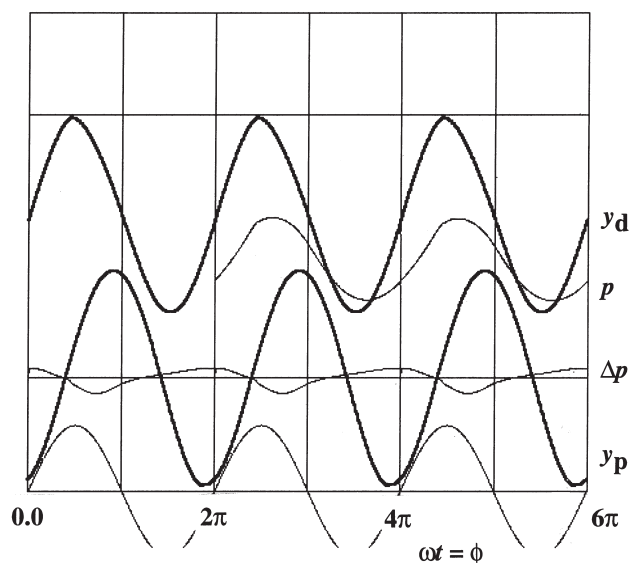


Fig. 9. Specimen motion diagram. Lowermost curve is driving voltage shown leading piston motion by about 45° . The bold trace above this represents the motion of the uppermost face of the piston. The light superimposed trace is pressure difference, $\Delta p = p_e - p_c$. The upper heavy trace follows the lowermost face of the displacer. The uppermost light trace shows expansion space pressure swing. Note that pressure difference is a sizeable fraction of pressure swing.

for achievement of the required phase between volume variations. At the same time it suggests where a search might be directed for a less inefficient way of realising the Stirling cycle in the free-piston embodiment.

7. Deductions

In overall terms, the computer models behaves in accordance with the supposed mechanism of operation: for example, increase of displacer drive rod diameter causes the displacer to over-travel, impacting the stop at the upper limit and/or the piston in the lower, as in Fig.

10(a). Excessive drive voltage, E_o , on the other hand, causes over-travel of the work piston (Fig. 10(b)). In both cases indicator diagrams become distorted, and traces of mass rate and Reynolds number reflect the sudden changes in flow direction. Raising flux density, B , across the airgap brings about a rise in electro-mechanical conversion efficiency, and calls for a higher driving voltage.

The model has not yet been subject to mechanised optimisation, but after a large number of manual perturbations of the variables, operation at the parameter values of Table 1 appears to be close to optimum (or to an optimum): computed heat lift at 77 K is 1.73 W for

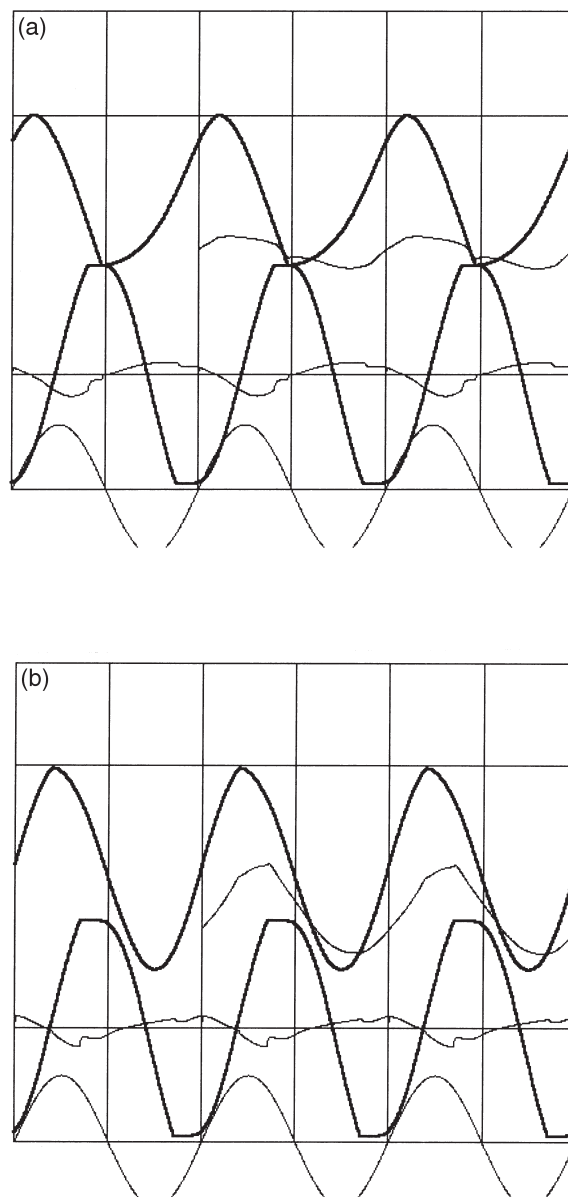


Fig. 10. Features of 'off-design' performance. Any mechanical contact appears to reduce heat lift capability and electro-mechanical conversion efficiency. (a) Displacer 'over-driven' by excessive value of d_{rod} . (b) Piston 'over-driven' by excessive input voltage.

a mechanical input of 29.37 W—or 16.97 W/W lifted, corresponding to a thermo-mechanical COP of 5.89%. Hymatic Engineering Co. claim [10] to lift 0.373 W from 82.2 K for 10.77 W (electrical) input, or 28.87 W/W nett in a somewhat smaller cooler (nominal 7 mm diameter cold-finger). To compete, the cooler of the present study would have to achieve an electro-mechanical conversion efficiency of $10.77/28.87 = 37.3\%$. In fact, the efficiency computed on the assumption of the 2.0 T(esla) flux density through the solenoid is only 22%, remediable only by an improved permanent magnet specification, and/or ironwork giving higher flux concentration.

The study affords less obvious insights into design and operation:

- A frequently-expressed aim of work to develop this type of device is enhancement of regenerator thermal performance. The finding here is that temperature recovery ratio is not an issue: the temperature difference between fluid stream and matrix attains its equilibrium value of 0.18°C rms within the first integration step of the computational sequence through the regenerator. Step size used here is standardised at 1/20 of regenerator length, so regenerator heat transfer capability is more than adequate, as confirmed by a high rms value of $NTU_r = 132$ at the rated conditions assumed. The massive value of thermal capacity ratio, N_{TCR} (1250 from Table 1) contributes to a high temperature recovery ratio.
- The expectation from experience with Stirling prime movers is for heat transfer in the working spaces to be poor. By contrast, rms temperature difference in the expansion space is 6.0°C; that in the compression space 13.5°C. Had this information been available at the time the NAX-106 cooler was designed [3], the compression-end heat exchanger would not have been incorporated, and dead space, compression ratio and overall performance would have benefited.
- The basic thermodynamic design problem is achievement of a displacer motion lagging that of the piston by a suitable amount. All other things being equal, displacer *amplitude* is controlled by drive rod diameter—with changes in buffer space dead volume also having a small effect—but the phase relationship is a result of a more subtle combination of factors. To appreciate these it is necessary to be clear that for present purposes the *natural frequencies*, ω_p and ω_d , of piston and displacer are those calculated in terms of respective masses, m_p and m_d , and the corresponding stiffnesses, k_p and k_d of the mechanical springs and *not* the pseudo-spring constants of the gas circuit of the ‘linearised’ approach.

Thus it is seen that increasing the mass of the displacer (say) whilst maintaining natural frequency (by increasing k_d in proportion) retards response to the pressure difference acting on the drive rod and

increases phase shift.

A decrease in hydraulic radius, r_h , of the wire mesh may be used to increase phase shift, but generally at a cost of increased pumping power. In a conventional Stirling engine or cooler, i.e. one with kinematically-synchronised piston and displacer, the increase would be thought of as an unacceptable performance penalty. A different interpretation applies here: the traces show a pressure *drop* to be a sizeable fraction of pressure *swing*. If this is the price to be paid for achieving a phase shift which in turn achieves heat lift, then it has to be paid. On the other hand, with the generous oversupply of regenerator heat transfer capacity suggesting an *increase* in hydraulic radius and corresponding *reduction* in pumping power, a search for performance improvement might focus on other parameters having an influence on phase shift.

- Electro-mechanical conversion efficiency is a strong function of operating conditions, and for the specification of Table 1 peaks at about 25%. The most influence parameter of conversion efficiency is flux density, B . The value assumed here is 2.0 (from the demagnetisation curve of neodymium–iron–boron) enhanced by a factor of 1.53 for the concentrating effect of the ironwork. Use of a higher value calls for increased driving voltage, E_o , but considerably enhances conversion efficiency. Where overall coefficient of performance (ratio of heat lifted to power input) is critical it appears important to embody a magnet material of high remanence and/or a high ratio of magnet cross-section to cross-section of the ironwork.
- For many combinations of design parameters it is possible to determine by trial and error a combination of input voltage, E_o , and angular frequency, ω ($= 2\pi f$) which will force a favourable phase-shift between piston and displacer, and a corresponding lift of heat from the expansion end. If development is approached in this way, the resulting heat lift rate (W) and overall COP are not under the control of the designer to the extent that they might be.

Availability of a computer simulation, by contrast, allows selection of the desired combination of design parameters and operating conditions in advance of manufacture.

Appendix A

Elementary view of the electro-magnetic circuit

Supplementary notation

B	magnetic flux density = $\mu_0 H$ in free space	Tesla
D	diameter of coil	m

d	diameter of wire of coil	m
E_o	amplitude (semi-swing) of supply voltage	V
e	emf	V
e_b	back emf	V
i	current	A
L_{ag}	length of air gap	m
L_c	overall physical length of coil	m
l	length of conductor	m
F	instantaneous force	N
H	magnetic field strength, or 'magnetising force'	amp/m
N	total number of turns in solenoid	
n	number of turns cut by flux path	
μ_0	permeability of free space $4\pi \times 10^{-7}$ Henry/m	Henry/m
μ_r	relative permeability—ratio of permeability of given material to that of free space: $B = \mu_r \mu_0 H$	
ρ	resistivity	ohm.m
ω	angular frequency	rad/s

Subscripts

ag	air gap
b	back (emf)
c	coil
sol	solenoid
w	wire

Principle of operation

Fig. 11(a) is a schematic representation of an electromagnetic circuit. This comprises a stationary magnetic

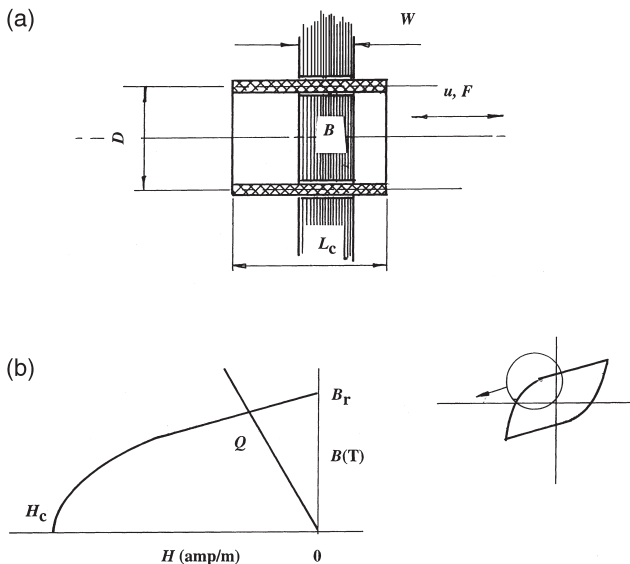


Fig. 11. Notation for electromagnetic circuit. (a) Solenoid with n effective turns cut by flux density B . (b) Part of demagnetisation curve of permanent magnet.

flux path energised by a permanent magnet and a coil wound into a rigid annulus. Permanent magnet and flux path are in the form of continuous annuli. The coil is rigidly attached to the work piston which is free to oscillate along the axis of the assembly. The axial length of the coil, L_c , exceeds that of the pole piece by an amount which ensures that the flux path cuts the same number of turns regardless of coil axial location. This condition confines coil excursions to the limits indicated.

Motion of the coil at velocity u , generates a back emf, e_b :

$$e_b = \pi B u D N W / L_c \quad (A1)$$

At the sort of cyclic frequencies envisaged, the corresponding instantaneous current is given by the ratio of net voltage to total coil resistance:

$$i = \frac{e - e_b}{R} \quad (A2)$$

Instantaneous axial force generated is proportional to current ($F = B i l$)

$$F_{sol} = \pi B i D N W / L_c \quad (A3)$$

It is necessary to have F_{sol} normalised to the same base as the equations of motion, i.e. by $k_p V_{sw}^{1/3} = k_p V_{sw}^{1/3}$. Assuming a supply voltage of the form $e = E_o \sin \omega t$, substituting Eq. (A2) into Eq. (A3) and normalising:

$$\frac{F_{sol}}{k_p V_{sw}^{1/3}} = \frac{\pi B D N W}{k_p V_{sw}^{1/3} L_c} \frac{1}{R} \{E_o \sin \omega t - \pi B u D n W / L_c\}$$

There is some economy of algebra in expressing coil resistance, R , in terms of resistivity of wire material and coil geometry:

$$R = \frac{\pi n D \rho}{1/4 \pi d^2} = 4 N D \rho / d^2 \quad (A4)$$

Making the substitution, and normalising u as per the thermodynamic treatment, i.e. by $\underline{u} = u / \omega L_{ref} = d\lambda_p / d\phi$, where $d\lambda_p$ is the current, instantaneous linear increment in piston position

$$\frac{F_{sol}}{k_p V_{sw}^{1/3}} = D G 1 \{ \sin \phi - D G 2 \underline{u} \} = F'_{sol} \quad (A5)$$

$$D G 1 = \frac{E_o B W \pi d^2}{4 \rho k_p V_{sw}^{1/3} L_c}$$

$$D G 2 = \frac{\pi B \omega L_{ref} D N W}{E_o L_c}$$

Determination of operating point

Fig. 11(b) represents part of the demagnetisation curve (B – H) curve of a permanent magnet. Over the range for which the curve is linear, hysteresis may be ignored, i.e. a shift in operating point, Q , caused by the demagnetising effect (air gap, ampere turns or an opposing magnetic field) may be reversed by removal of the demagnetising effect. Point Q is determined in terms of the B and H allowed by the flux circuit of which the permanent magnet is part.

Law of flux continuity

In Fig. 12(a) the flux, ϕ , which links the permanent magnet passes through the soft-iron circuit and the air gap

$$\phi = B_{\text{magnet}} A_{x_{\text{magnet}}} = B_{\text{iron}} A_{x_{\text{iron}}} = B_{\text{air}} A_{x_{\text{air}}} \quad (\text{A6})$$

Ampere's circuital law, $\int \underline{H} \cdot d\underline{l} = ni$

Applying the law to Fig. 12(b)

$$H_{\text{iron}} L_{\text{iron}} + H_{\text{air}} L_{\text{ag}} = ni \quad (\text{A7})$$

In the annular iron path shown $L_{\text{iron}} = \pi D - g \approx \pi D$. For the special case of $ni = 0$ ($n = 0$ and/or $i = 0$) consideration of remanent fields leads to $H_{\text{iron}} \pi D = -H_{\text{air}} L_{\text{ag}}$.

In Fig. 12(b) there is no current-carrying coil, so $ni = 0$ and $H_{\text{magnet}} L_{\text{magnet}} + H_{\text{iron}} L_{\text{iron}} + H_{\text{air}} L_{\text{ag}} = 0$. Taking account of Eq. (A6), (A7), and using $\mu_0 \mu_r^{\text{air}} = \mu_{\text{air}} \ll \mu_0 \mu_r^{\text{iron}}$ gives

$$B_{\text{magnet}}/H_{\text{magnet}} = -\mu_0 (A_{x_{\text{air}}}/A_{x_{\text{magnet}}}) (L_{\text{magnet}}/L_{\text{ag}}) \quad (\text{A8})$$

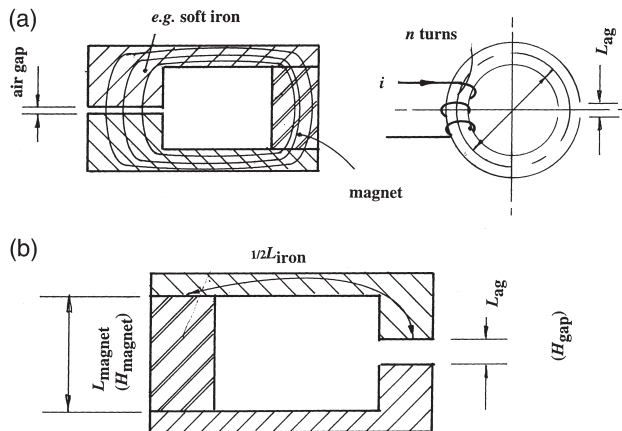


Fig. 12. Determination of operating point in terms of circuit geometry: (a) ironwork circuit; (b) air gap and ironwork as magnetic circuit elements.

Eq. (A8) plots into Fig. 11(b) as a straight line of negative gradient, determining operating point Q .

Appendix B

Preparation of gas process equations for solution

When Eqs. (6) and (8) are written for each of the two working spaces, the resulting total of six equations are, but for the U terms, linear in the six unknowns, dm_e , dm_c , dp_e , dT_{ge} , dT_{gc} and dp_c (in that order).

For generality, computation proceeds in terms of normalised variables and dimensionless parameters. Eq. (2) is written out for each working space in turn:

$$\begin{aligned} & NTU_e (\sigma_e / \delta_e) (N_T - \tau_{ge}) d\phi + d\sigma_e \{ \tau_{ge} [1 \\ & - U(d\sigma_e)] + U(d\sigma_e) N_T \} - \psi_e d\delta_e - \tau_{ge} d\sigma_e \\ & + \sigma_e d\tau_{ge} = 0 \end{aligned} \quad (\text{B1a})$$

$$\begin{aligned} & NTU_c (\sigma_c / \delta_c) (1 - \tau_{gc}) d\phi + d\sigma_c \{ \tau_{gc} [1 \\ & - U(d\sigma_c)] + U(d\sigma_c) \} - \psi_c d\delta_c - \tau_{gc} d\sigma_c \\ & + \sigma_c d\tau_{gc} = 0 \end{aligned} \quad (\text{B1b})$$

The NTU are respective 'number of transfer units', typified by:

$$NTU_e = \frac{a R_e^{b-1} S_e}{\pi N_{pr} L_{ref}}$$

Normalised mass balance is

$$d\sigma_e + d\sigma_c + \nu_D 1/2 (d\psi_e + d\psi_c) = 1 \quad (\text{B2})$$

The normalised description is completed by writing the difference form of the gas law for each working space:

$$\tau_{ge} d\sigma_e + \sigma_e d\tau_{ge} - \delta_e d\psi = \psi_e d\delta_e \quad (\text{B3a})$$

$$\tau_{gc} d\sigma_c + \sigma_c d\tau_{gc} - \delta_c d\psi = \psi_c d\delta_c \quad (\text{B3b})$$

The integration process involves writing the coefficients of the six unknowns $d\sigma_e$, $d\sigma_c$, $d\psi_e$, $d\tau_{ge}$ and $d\tau_{gc}$ and $d\psi_e$ (in that sequence) as a 6×6 matrix and solving numerically. Non-zero coefficients of unknowns in the order just given are:

Energy-expansion end

$$a(1,1) = \tau_{ge} (1 - U(d\sigma_e) - 1/\gamma) + U(d\sigma_e) \tau_{roe}$$

$$a(1,4) = -1/2 NTU_e \Delta\phi - \sigma_e / \gamma$$

$$b(1) = -NTU_e (N_T - \tau_{ge}) \Delta\phi + \psi_e d\nu_e (\gamma - 1) / \gamma$$

Energy-compression end

$$a(2,2) = \tau_{gc}(1 - U(d\sigma_c) - 1/\gamma) + U(d\sigma_c)\tau_{roc}$$

$$a(2,5) = -1/2NTU_c\alpha\Delta\phi - \sigma_c/\gamma$$

$$b(2) = -NTU_c(1.0 - \tau_{gc})\Delta\phi + \psi_c d\nu_c(\gamma - 1)/\gamma$$

The τ_{roe} and τ_{roc} are the appropriate regenerator outlet temperatures computed with the aid of Eq. (15) of the main text.

Mass conservation

$$a(3,1) = 1$$

$$a(3,2) = 1$$

$$a(3,3) = 1/2\nu_D$$

$$a(3,6) = 1/2\nu_D$$

Gas law—expansion end

$$a(4,2) = \tau_{\geq}$$

$$a(4,3) = -\nu_e$$

$$a(4,4) = \sigma_e$$

$$b(4) = \psi_e d\delta_e$$

Gas law—compression end

$$a(5,2) = \tau_{gc}$$

$$a(5,3) = -\nu_c$$

$$a(5,5) = \sigma_c$$

$$b(5) = \psi_c d\delta_c$$

Momentum

$$a(6,1) = - (d\sigma_e \Sigma_1 + d\sigma_c \Sigma_3)DGM$$

$$a(6,2) = - (d\sigma_c \Sigma_2 + d\sigma_e \Sigma_3)DGM$$

$$a(6,3) = 1/2$$

$$a(6,6) = -1/2$$

$$b(6) = -\psi_e + \psi_c$$

$$DGM = \frac{dL_{reg}}{L_{ref}} \frac{N_{MA}^2}{\alpha_{ffr}^2} \frac{L_{reg}}{r_h} \frac{1}{\Delta\phi^2} \frac{1}{\psi_e + \psi_c}$$

The various Σ_n are the normalised equivalents of the Σ_n of Eq. (12).

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